EQUATION OF THE ELASTOPLASTIC MODE OF FILTRATION

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We consider the Cauchy problem for the equation of the elastoplastic mode of filtration. We prove that a solution in the form of an instantaneous source function cannot be obtained for this equation in a natural manner.

We turn our attention to the equation

$$\frac{\partial^2 u}{\partial x^2} = \alpha \frac{\partial u}{\partial t} + \beta \left| \frac{\partial u}{\partial t} \right| \quad (\alpha > 0, |\beta| < \alpha) \tag{1}$$

arising in the course of description of the elastoplastic mode of filtration [1, 2] and consider the solution of the Cauchy problem for this equation

$$u|_{t=0} = \varphi(x) \tag{2}$$

in the strip R_T $(|x| < \infty, 0 \leqslant t \leqslant T)$ of arbitrary, fixed width T.

In [3] it was proved that if the function $\varphi(x)$ is bounded on the whole straight line and has a generalized derivative $\varphi'(x) \in L_2(-\infty, \infty)$, there exists a unique continuous solution $u(x, t)$ of (1) bounded in R_T, satisfying the condition (2) and possessing continuous derivatives $\partial u/\partial t$, $\partial u/\partial x$ and $\partial^2 u/\partial x^2$ for $t > 0$.

In the present paper we show that a solution of (1) in form of an instantaneous source function, i.e. a solution satisfying the condition

$$u|_{t=0} = \delta(x) \tag{3}$$

cannot be constructed.

Below we consider delta-form sequences of smooth functions. We shall say that the sequence $\varphi_k(x)$ converges to $\delta(x)$, if

$$\int_{-\infty}^{\infty} \varphi_k(x) f(x)\, dx \to f(0)$$

holds for any infinitely differentiable function $f(x)$ decreasing with $|x| \to \infty$ faster than $\exp(-mx^2)$ for any arbitrary m.

For $\varphi_k(x) \to \delta(x)$ it is sufficient to have

$$\lim_{k\to\infty} \int_a^b \varphi_k(x)\, dx = 1, \quad \lim_{k\to\infty} \int_b^\infty \varphi_k(x)\, dx = \lim_{k\to\infty} \int_{-\infty}^a \varphi_k(x)\, dx = 0 \tag{4}$$

for any fixed $a < 0$ and $b > 0$.

Theorem 1. Let the functional sequence $\varphi_k(x)$ satisfy the conditions (4). We assume that each function $\varphi_k(x) \geqslant 0$ is bounded on the whole straight line and has the derivative $\varphi_k'(x) \in L_2(-\infty, \infty)$.

Then, for any k there exists a continuous solution $u_k(x, t)$ of (1) bounded in R_T and satisfying the condition

$$u_k|_{t=0} = \varphi_k(x) \tag{5}$$

and we have

$$\lim_{k\to\infty} u_k(x_0, t_0) = 0 \quad (\beta > 0), \quad \lim_{k\to\infty} u_k(x_0, t_0) = \infty \quad (\beta < 0)$$

at any point $(x_0, t_0) \in R_T$.

We shall need several auxiliary lemmas for the proof of this theorem.

Lemma 1. Let $u_1(x,t)$ and $u_2(x, t)$ be two continuous solutions of (1) bounded in R_T and satisfying the respective conditions

$$u_1|_{t=0} = \varphi_1(x), \quad u_2|_{t=0} = \varphi_2(x)$$

and let continuous derivatives $\partial u_i / \partial t$ and $\partial^2 u_i / \partial x^2$ $(i = 1, 2)$ exist for $t > 0$. Then

$$u_1(x, t) \leqslant u_2(x, t) \text{ for } \varphi_1(x) \leqslant \varphi_2(x)$$

Proof. Let $|u_1(x, t)| \leqslant M$ and $|u_2(x, t)| \leqslant M$ in R_T. We shall consider the region $\Omega(|x| \leqslant L,\ 0 \leqslant t \leqslant T)$ where L is a number which, in the following, shall be gradually increased. We set

$$F(x, t) = \frac{4M}{L^2}\left(\frac{x^2}{2} + \frac{1}{\alpha - |\beta|}t\right), \quad z(x, t) = [F(x, t) - u_1(x, t) + u_2(x, t)]\, e^{-\lambda t} \quad (\lambda > 0)$$

Function $z(x, t)$ satisfies the equation

$$(\alpha - |\beta|)\frac{\partial z}{\partial t} + \lambda(\alpha - |\beta|)z - \frac{\partial^2 z}{\partial x^2} = $$
$$= |\beta| e^{-\lambda t}\left[\frac{\partial(u_1 - u_2)}{\partial t} + \operatorname{sign}\beta\left(\left|\frac{\partial u_1}{\partial t}\right| - \left|\frac{\partial u_2}{\partial t}\right|\right)\right] \quad (6)$$

and the latter implies that $z(x, t)$ cannot have a negative minimum within Ω. Indeed, the function in the square brackets and $\partial(u_1 - u_2)/\partial t$ have the same sign. But

$$\frac{\partial(u_1 - u_2)}{\partial t} = \frac{\partial F}{\partial t} - \frac{\partial z}{\partial t}e^{\lambda t} - \lambda z e^{\lambda t} > 0$$

Therefore, the point of negative minimum in (6) should have a positive function on its right and a negative one on its left, which is clearly not possible. In addition, $z(x, t)$ satisfies the conditions

$$z(x, 0) \geqslant 0, \qquad z(L, t) \geqslant 0, \qquad z(-L, t) \geqslant 0$$

It therefore follows that $z(x, t) \geqslant 0$ in Ω and consequently

$$u_1(x, t) - u_2(x, t) \leqslant \frac{4M}{L^2}\left(\frac{x^2}{2} + \frac{1}{\alpha - |\beta|}t\right) \qquad (x, t) \in \Omega$$

Passing to the limit as $L \to \infty$ we find that at each fixed point (x, t) we have

$$u_1(x, t) - u_2(x, t) \leqslant 0$$

Lemma 2. Let $u(x, t)$ be a continuous solution of (1) satisfying the condition (2), bounded in R_T. $v(x, t)$ a continuous solution of

$$\frac{\partial^2 v}{\partial x^2} = (\alpha + \beta)\frac{\partial v}{\partial t} \quad (7)$$

bounded in R_T and satisfying the condition

$$v|_{t=0} = \varphi_1(x) \quad (8)$$

and $w(x, t)$ a continuous solution of

$$\frac{\partial^2 w}{\partial x^2} = (\alpha - \beta)\frac{\partial w}{\partial t} \quad (9)$$

bounded in R_T and satisfying the condition

$$w|_{t=0} = \varphi_1(x) \quad (10)$$

We assume that the functions $u(x, t)$, $v(x, t)$ and $w(x, t)$ have continuous derivatives appearing in the equations. Then

$$u(x, t) \leqslant v(x, t), \quad u(x, t) \leqslant w(x, t) \quad \text{for } \beta > 0,\ \varphi(x) \leqslant \varphi_1(x) \quad (11)$$
$$u(x, t) \geqslant v(x, t), \quad u(x, t) \geqslant w(x, t) \quad \text{for } \beta < 0,\ \varphi(x) \geqslant \varphi_1(x) \quad (12)$$

Proof. Consider the case $\beta > 0$ and set $v_1(x, t) = u(x, t) - v(x, t)$. We clearly have

$$\frac{\partial^2 v_1}{\partial x^2} - (\alpha+\beta)\frac{\partial v_1}{\partial t} = \beta\left(\left|\frac{\partial u}{\partial t}\right| - \frac{\partial u}{\partial t}\right) \geqslant 0$$

$$v_1 |_{t=0} = \varphi(x) - \varphi_1(x) \leqslant 0$$

From this it follows that the maximum principle ([4], Theorem 8, Sect. 1) is valid for the function $v_1(x, t)$. According to this principle, $v_1(x, t) \leqslant 0$. Setting now $w_1(x, t) = u(x, t) - w(x, t)$ we obtain

$$\frac{\partial^2 w_1}{\partial x^2} - (\alpha-\beta)\frac{\partial w_1}{\partial t} = \beta\left(\left|\frac{\partial u}{\partial t}\right| + \frac{\partial u}{\partial t}\right) \geqslant 0$$

$$w_1 |_{t=0} = \varphi(x) - \varphi_1(x) \leqslant 0$$

Consequently, the maximum principle holds also for $w_1(x, t)$, i. e. $w_1(x, t) \leqslant 0$. This yields both inequalities (11). Inequalities (12) can be proved in a similar manner.

Let us now denote by $V(x, t, \xi)$ the basic solution of (7) and by $W(x, t, \xi)$ the basic solution of (9). We know that

$$V(x, t, \xi) = \frac{\sqrt{\alpha+\beta}}{2\sqrt{\pi t}} \exp\frac{-(x-\xi)^2(\alpha+\beta)}{4t}$$

$$W(x, t, \xi) = \frac{\sqrt{\alpha-\beta}}{2\sqrt{\pi t}} \exp\frac{-(x-\xi)^2(\alpha-\beta)}{4t}$$

Let us set

$z(x, t) = \min[V(x, t, 0), W(x, t, 0)]$, $z_1(x, t) = \max[V(x, t, 0), W(x, t, 0)]$

and write the following properties of these functions which we shall need later

$$\int_{-\infty}^{\infty} z(x, t)\, dx = B < 1, \quad \int_{-\infty}^{\infty} z_1(x, t)\, dx = B_1 > 1 \quad (t > 0) \tag{13}$$

Here B and B_1 are independent of t, but depend on the values of α and β. Let us verify e. g. the first of these inequalities

$$\frac{1}{2\sqrt{\pi t}} \int_{-\infty}^{\infty} \min\left[\sqrt{\alpha+\beta}\exp\frac{-x^2(\alpha+\beta)}{4t},\ \sqrt{\alpha-\beta}\exp\frac{-x^2(\alpha-\beta)}{4t}\right] dx =$$

$$= \frac{1}{2\sqrt{\pi}} \int_{-\infty}^{\infty} \min\left[\sqrt{\alpha+\beta}\exp\frac{-y^2(\alpha+\beta)}{4},\ \sqrt{\alpha-\beta}\exp\frac{-y^2(\alpha-\beta)}{4}\right] dy =$$

$$= \frac{1}{2\sqrt{\pi}} \int_{-\infty}^{\infty} \min[f_1(y), f_2(y)]\, dy = B \tag{14}$$

The integrand function is independent of t, hence $B = \text{const}$. Moreover,

$$\frac{1}{2\sqrt{\pi}} \int_{-\infty}^{\infty} f_1(y)\, dy = \frac{1}{2\sqrt{\pi}} \int_{-\infty}^{\infty} f_2(y)\, dy = 1$$

therefore

$$\frac{1}{2\sqrt{\pi}} \int_{-\infty}^{\infty} \min[f_1(y), f_2(y)]\, dy < 1$$

L e m m a 3. Let the sequence of bounded functions $\psi_k(x) \geqslant 0$ satisfy the conditions (4). Further, let the function $v_k(x, t)$ satisfy Eq. (7) and the condition

$$v_k|_{t=0} = \psi_k(x)$$

and the function $w_k(x, t)$ satisfy Eq. (9) and the condition

$$w_k|_{t=0} = \psi_k(x)$$

Then for any fixed $t^* > 0$ and any $\varepsilon > 0$ there exists a k_0 such that if $k \geqslant k_0$, the following inequalities are valid for all x and for $t \in [t^*, t^* + 1]$:

$$(1 - \varepsilon) V(x, t, 0) - \varepsilon \leqslant v_k(x, t) \leqslant (1 + \varepsilon) V(x, t, 0) + \varepsilon \qquad (15)$$

$$(1 - \varepsilon) W(x, t, 0) - \varepsilon \leqslant w_k(x, t) \leqslant (1 + \varepsilon) W(x, t, 0) + \varepsilon \qquad (16)$$

Proof. We know that

$$v_k(x, t) = \int_{-\infty}^{\infty} \psi_k(\xi) V(x, t, \xi)\, d\xi$$

hence

$$v_k(x, t) = \int_{-\delta}^{\delta} \psi_k(\xi) V(x, t, \xi)\, d\xi + \int_{-\infty}^{-\delta} \psi_k(\xi) V(x, t, \xi)\, d\xi +$$

$$+ \int_{\delta}^{\infty} \psi_k(\xi) V(x, t, \xi)\, d\xi = I_1^k + I_2^k + I_3^k$$

where the constant δ $(0 < \delta < 1)$ will be chosen.

Let us consider the integral

$$I_1^k(x, t) = \int_{-\delta}^{\delta} \psi_k(\xi) V(x, t, \xi)\, d\xi$$

We choose the quantity A large enough to ensure that $V(x, t, \xi) < {}^1/_4\varepsilon$ when $|x| \geqslant A$, $|\xi| < 1$, and $t \in [t^*, t^* + 1]$; then we choose δ sufficiently small for the inequalities

$$1 - \frac{\varepsilon}{2} < \frac{V(x, t, \xi)}{V(x, t, 0)} = \exp \frac{(2x\xi - \xi^2)(\alpha + \beta)}{4t} < 1 + \frac{\varepsilon}{2}$$

to hold for

$$|\xi| \leqslant \delta, \ |x| \leqslant A, \ t \in [t^*, t^* + 1]$$

Then

$$(1 - \varepsilon) V(x, t, 0) - \varepsilon < 0 \leqslant I_1^k(x, t) < \frac{\varepsilon}{4} \int_{-\delta}^{\delta} \psi_k(\xi)\, d\xi$$

at $|x| \geqslant A$, $t \in [t^*, t^* + 1]$

$$\left(1 - \frac{\varepsilon}{2}\right) V(x, t, 0) \int_{-\delta}^{\delta} \psi_k(\xi)\, d\xi < I_1^k(x, t) < \left(1 + \frac{\varepsilon}{2}\right) V(x, t, 0) \int_{-\delta}^{\delta} \psi_k(\xi)\, d\xi$$

at $|x| \leqslant A$, $t \in [t^*, t^* + 1]$

Since the sequence $\psi_k(\xi)$ satisfies the conditions (4), we have

$$\int_{-\delta}^{\delta} \psi_k(\xi)\, d\xi \to 1$$

This, together with the preceding inequalities implies that there exists k_1 such, that

$$(1 - \varepsilon) V(x, t, 0) - \varepsilon < I_1^k(x, t) < (1 + \varepsilon) V(x, t, 0) + {}^1/_2\varepsilon \qquad (k \geqslant k_1)$$

Since $v_k(x, t) \geqslant I_1^k(x, t)$, we have

$$v_k(x, t) \geqslant (1 - \varepsilon) V(x, t, 0) - \varepsilon$$

From the conditions (4) as well as from the boundedness of the function $V(x, t, 0)$ with $t \in [t^*, t^*+1]$, and for all x and ξ, it follows that

$$I_2^k(x, t) + I_3^k(x, t) \to 0 \quad \text{as} \quad k \to \infty$$

We can therefore find $k_0 \geqslant k_1$ such, that for $t \in [t^*, t^*+1]$,

$$v_k(x, t) = I_1^k + I_2^k + I_3^k \leqslant (1+\varepsilon) V(x, t, 0) + \varepsilon \qquad (k \geqslant k_0)$$

This proves both inequalities (15). The estimates (16) can be proved in the similar manner.

Corollary. Fulfillment of the conditions of Lemma 3 means that for any fixed $t^* > 0$ and any $\varepsilon > 0$ there exists a k_0 such that if $k \geqslant k_0$, the following inequalities are valid for all x and for $t \in [t^*, t^*+1]$:

$$\min [v_k(x, t), w_k(x, t)] \leqslant (1+\varepsilon) z(x, t) + \varepsilon$$
$$\max [v_k(x, t), w_k(x, t)] \geqslant (1-\varepsilon) z_1(x, t) - \varepsilon$$

We shall now pass to the proof of Theorem 1. Let $\varphi_k(x)$ be a sequence of functions mentioned in the conditions of the theorem. Then, as shown in [3], for any k there exists a unique solution $u_k(x, t)$ of (1), satisfying (5), which is bounded in R_T and has continuous derivatives $\partial u_k / \partial t$, $\partial u_k / \partial x$ and $\partial^2 u_k / \partial x^2$ when $t > 0$. From Lemma 1 it follows that $u_k(x, t) \geqslant 0$.

Let some value t_0 be fixed. Let us choose t_1 such, that $t_1 < t_0 < t_1 + 1$.

We shall first consider the case when $\beta > 0$.

We shall prove, that for any integral value of n and for any $\varepsilon \in [0, B^{-1} - 1]$ such $k_0(\varepsilon, n)$ and $t_2 \in [t_1, t_0]$ can be found, that for $k \geqslant k_0(\varepsilon, n)$ the following inequality is valid

$$u_k(x, t_0) \leqslant [B(1+\varepsilon)]^n (1+\varepsilon) z(x, t_2) + (n+1)\varepsilon, \quad |x| < \infty \qquad (17)$$

Proof of the theorem for the case $\beta > 0$ follows from this inequality, as the function $z(x, t_2)$ is bounded for all $t_2 \in [t_1, t_0]$ and $|x| < \infty$, while the constants ε and n can be chosen, so as to make the quantities $[B(1+\varepsilon)]^n$ and $(n+1)\varepsilon$ as small as we want ($B < 1$ by (13)).

Let us now denote by $v_k(x, t)$ and $w_k(x, t)$ the respective solutions of (7) and (9), satisfying the conditions

$$v_k(x, 0) = w_k(x, 0) = \varphi_k(x)$$

Let $\{\tau_i\}$ be an infinite sequence of positive numbers tending to zero and $\tau_1 + \tau_2 + \tau_3 + \ldots < t_0 - t_1$. We set $\psi_i(x) = B^{-1} z(x, \tau_i)$. We can easily confirm that conditions (4) hold for the functional sequence $\psi_i(x)$, i.e. $\psi_i(x) \to \delta(x)$. Denoting by $v_i^*(x, t)$ and $w_i^*(x, t)$ the respective solutions of (7) and (9) satisfying the conditions

$$v_i^*(x, 0) = w_i^*(x, 0) = \psi_i(x)$$

we shall construct a subsequence τ_{i_m} $(1 \leqslant m \leqslant n)$ in the following manner. We shall choose i_1 sufficiently large for

$$\min [v_{i_1}^*(x, t), w_{i_1}^*(x, t)] \leqslant (1+\varepsilon) z(x, t) + \varepsilon \text{ when } \in [t_1, t_1+1] \qquad (18)$$

to hold. Such i_1 exists by the corollary to Lemma 3. Further we choose $i_2 > i_1$ sufficiently large for

$$\min [v_{i_2}^*(x, \tau_{i_1}), w_{i_2}^*(x, \tau_{i_1})] \leqslant (1+\varepsilon) z(x, \tau_{i_1}) + \varepsilon$$

to hold, and $i_3 > i_2$ such that

$$\min [v_{i_3}^*(x, \tau_{i_2}), w_{i_3}^*(x, \tau_{i_2})] \leqslant (1+\varepsilon) z(x, \tau_{i_2}) + \varepsilon$$
$$\cdots\cdots\cdots\cdots\cdots\cdots\cdots\cdots\cdots\cdots$$
$$\min [v_{i_n}^*(x, \tau_{i_{n-1}}), w_{i_n}^*(x, \tau_{i_{n-1}})] \leqslant (1+\varepsilon) z(x, \tau_{i_{n-1}}) + \varepsilon \qquad (19)$$

Having thus reached i_n, we shall select the number k_0 sufficiently large for

$$\min[v_k(x, \tau_{i_n}),\ w_k(x, \tau_{i_n})] \leqslant (1+\varepsilon) z(x, \tau_{i_n}) + \varepsilon \qquad (k > k_0)$$

to hold. Let us consider the functions $u_k(x, t)$ and $k \geqslant k_0$. By Lemma 2 we have, for all x and t

$$u_k(x, t) \leqslant \min[v_k(x, t),\ w_k(x, t)]$$

Setting $t = \tau_{i_n}$, we obtain (20)

$$u_k(x, \tau_{i_n}) \leqslant \min[v_k(x, \tau_{i_n}), w_k(x, \tau_{i_n})] \leqslant (1+\varepsilon) z(x, \tau_{i_n}) + \varepsilon \quad (k > k_0)$$

Clearly, the function $B(1+\varepsilon) v_{i_n}^*(x, t) + \varepsilon$ satisfies Eq. (7) and condition

$$B(1+\varepsilon) v_{i_n}^*(x, 0) + \varepsilon = (1+\varepsilon) z(x, \tau_{i_n}) + \varepsilon$$

while the function $B(1+\varepsilon) w_{i_n}^*(x, t) + \varepsilon$ satisfies (9) and condition

$$B(1+\varepsilon) w_{i_n}^*(x, 0) + \varepsilon = (1+\varepsilon) z(x, \tau_{i_n}) + \varepsilon$$

Therefore from (20) and Lemma 2 it follows that for all x and $t > 0$

$$u_k(x, \tau_{i_n} + t) \leqslant \min[B(1+\varepsilon) v_{i_n}^*(x, t) + \varepsilon, B(1+\varepsilon) w_{i_n}^*(x, t) + \varepsilon] = \\ = B(1+\varepsilon) \min[v_{i_n}^*(x, t), w_{i_n}^*(x, t)] + \varepsilon$$

Setting $t = \tau_{i_{n-1}}$ we have

$$u_k(x, \tau_{i_n} + \tau_{i_{n-1}}) \leqslant B(1+\varepsilon) \min[v_{i_n}^*(x, \tau_{i_{n-1}}), w_{i_n}^*(x, \tau_{i_{n-1}})] + \varepsilon$$

which together with (19) yields

$$u_k(x, \tau_{i_n} + \tau_{i_{n-1}}) \leqslant B(1+\varepsilon)[(1+\varepsilon) z(x, \tau_{i_{n-1}}) + \varepsilon] + \varepsilon = \\ = B(1+\varepsilon)^2 z(x, \tau_{i_{n-1}}) + B(1+\varepsilon)\varepsilon + \varepsilon \leqslant B(1+\varepsilon)^2 z(x, \tau_{i_{n-1}}) + 2\varepsilon$$

The last inequality follows from the fact that $B(1+\varepsilon) \leqslant 1$.

Similarly, we can satisfy ourselves that

$$u_k(x, \tau_{i_n} + \tau_{i_{n-1}} + \tau_{i_{n-2}}) \leqslant B^2 (1+\varepsilon)^3 z(x, \tau_{i_{n-2}}) + 3\varepsilon$$

Continuing this process we obtain

$$u_k(x, \tau_{i_n} + \tau_{i_{n-1}} + \ldots + \tau_{i_1}) \leqslant B^{n-1}(1+\varepsilon)^n z(x, \tau_{i_1}) + n\varepsilon$$

which together with (18) yields

$$u_k(x, \tau_{i_n} + \tau_{i_{n-1}} + \ldots + \tau_{i_1} + t) \leqslant \\ \leqslant B^n (1+\varepsilon)^n \min[v_{i_1}^*(x, t),\ w_{i_1}^*(x, t)] + n\varepsilon \leqslant \\ \leqslant B^n (1+\varepsilon)^n [(1+\varepsilon) z(x, t) + \varepsilon] + n\varepsilon \leqslant \\ \leqslant B^n (1+\varepsilon)^{n+1} z(x, t) + (n+1)\varepsilon \quad t \in [t_1, t_1 + 1] \tag{21}$$

Let us put now $t_2 = t_0 - (\tau_{i_1} + \ldots + \tau_{i_n})$. Since $\tau_1 + \tau_2 + \tau_3 + \ldots < t_0 - t_1$,

$$t_0 - (\tau_{i_1} + \tau_{i_2} + \ldots + \tau_{i_n}) > t_0 - (t_0 - t_1) = t_1$$

Moreover we have $t_2 < t_0 < t_1 + 1$, consequently the estimate (21) can be used when $t = t_2$. Thus we have, for $k \geqslant k_0$,

$$u_k(x, t_0) \leqslant (1+\varepsilon)^{n+1} B^n z(x, t_2) + (n+1)\varepsilon$$

As was shown before, proof of the theorem for the case $\beta > 0$ follows from the latter inequality.

The theorem is proved for the case $\beta < 0$ in the analogous manner and the following assertion holds. For any integral n and for any $\varepsilon \in [0, 1 - B_1^{-1}]$ such $k_1(\varepsilon, n)$ and

$t_2 \in [t_1, t_0]$ can be found, that the inequality

$$u_k(x, t_0) \geqslant [B_1(1-\varepsilon)]^n (1-\varepsilon) z(x, t_2) - [B_1(1-\varepsilon)]^n \varepsilon (n+1) \qquad (22)$$

holds for all x and $k \geqslant k_1(\varepsilon, n)$.

Proof of this inequality is performed as the proof of (17). Relation (22) then yields the proof of the theorem of the case $\beta < 0$.

Indeed, $z(x_0, t_2) > q > 0$ for a fixed x_0 and for all $t_2 \in [t_1, t_0]$. Therefore, from (22) we obtain $u_k(x_0, t_0) \geqslant [B_1(1-\varepsilon)]^n [(1-\varepsilon)q - (n+1)\varepsilon]$

Let M be any constant and let us choose n sufficiently large to ensure that $[B_1(1-\varepsilon)]^n$ is greater than M for any $\varepsilon < {}^1/_2\,(1 - B_1^{-1})$. Such n exists because for all such ε we have $B_1(1-\varepsilon) > B_1(1 - {}^1/_2 + {}^1/_2 B_1^{-1}) = {}^1/_2 (B_1 + 1) > 1$

Further, keeping n fixed we choose ε sufficiently small for the following two inequalities to hold $\varepsilon < {}^1/_2\,(1 - B_1^{-1})$, $(1-\varepsilon)q - (n+1)\varepsilon > {}^1/_2 q$

With such values of n and ε we obtain

$$u_k(x_0, t_0) > {}^1/_2\, qM \quad [k \geqslant k_1(\varepsilon, n)]$$

Since M has been chosen arbitrarily,

$$\lim_{k\to\infty} u_k(x_0, t_0) = \infty$$

which completes the proof of Theorem 1.

Theorem 2 cited below asserts, that no function $u(x, t)$ exists which could be called a solution of (1) under the condition (3). We can naturally assume that if such function exists, it should be positive, continuous, and should have continuous derivatives for $t > 0$, and should satisfy Eq. (1). Let $t_k \to 0$ as $k \to \infty$ and let $\varphi_k(x) = u(x, t_k)$. We can expect by virtue of the equation $u(x, 0) = \delta(x)$, that the functional sequence $\varphi_k(x)$ satisfies the conditions (4).

Theorem 2. No positive function $u(x, t)$ exists, possessing the following properties:

1) its continuous derivatives $\partial u / \partial t$, $\partial u / \partial x$ and $\partial^2 u / \partial x^2$ exist when $t > 0$;
2) function $u(x, t)$ satisfies Eq. (1);
3) function $u(x, t)$ is bounded in any strip $(0 < \gamma \leqslant t \leqslant T)$;
4) if $t_k \to 0$ as $k \to \infty$, then conditions (4) hold for the functional sequence $\varphi_k(x) = u(x, t_k)$.

Proof. Assume that such function $u(x, t)$ exists. Then, for any k, the function $u_k(x, t) = u(x, t + t_k)$ is positive, continuous, bounded in R_T and satisfies Eq. (1). Clearly we have

$$u_k(x, 0) = u(x, t_k) = \varphi_k(x)$$

The functional sequence $\varphi_k(x)$ satisfies the conditions (4). Therefore all conditions of Theorem 1 bar $\varphi_k'(x) \in L_2(-\infty, \infty)$, are fulfilled. But this condition was used in the course of proving Theorem 1 only to establish the existence of a solution of (1), which would assume the values $\varphi_k(x)$ at $t = 0$. In the present case the existence of such solution is apparent, namely: $u_k(x, t) = u(x, t + t_k)$. Consequently we can apply Theorem 1 to the functional sequence $u_k(x, t)$. Let (x_0, t_0) be a fixed point belonging to R_T. Then

$$\lim_{t_k\to 0} u(x_0, t_0 + t_k) = \lim_{k\to\infty} u_k(x_0, t_0) = \begin{cases} 0 & (\beta > 0) \\ \infty & (\beta < 0) \end{cases}$$

This together with the continuity of $u(x, t)$ implies that for $(x_0, t_0) \in R_T$

$$u_k(x, 0) = u(x, t_k) = \varphi_k(x)$$

which contradicts the previous assumption, Q. E. D.

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UNSTEADY WAVES IN A ROTATING CHANNEL OF CONSTANT DEPTH

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Propagation of waves caused by the original rise on the surface of rotating liquid is considered. The deformation of the disturbed level proceeds in accordance with the theory of long waves. The unsteady part of the wave rise may be treated as a limit superposition of standing waves with phases allowing for a complete range of wave numbers. The original perturbations are assumed to act in such a manner that, as the distance between the nodes decreases, the elementary crests of an arbitrary component take a near-equilibrium position. It is permissible to use in such analysis of unsteady wave problems a Fourier integral the complex amplitude of which must be determined. Our analysis of waves in a channel is based on Sretenskii's general hydrodynamic analysis of tsunami waves on a rotating half-plane [1].

1. The value of the perturbed level in a channel of constant depth h can be found from the wave equation

$$\frac{\partial^2 \zeta}{\partial t^2} + 4\omega^2 \zeta = gh\left(\frac{\partial^2 \zeta}{\partial x^2} + \frac{\partial^2 \zeta}{\partial y^2}\right) \tag{1.1}$$

in which the initial functions are

$$\zeta(x, y, 0) = M(x, y), \quad \partial\zeta(x, y, 0) / \partial t = N(x, y) \tag{1.2}$$

those perturbed motions that do not result variations of the liquid level in time are not considered.

Assuming shockless initial values of the transverse components of the velocity and acceleration, the conditions of impenetrability at the boundaries $y = 0$, $y = l$ can